NOTES AND CORRESPONDENCE

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Numerical Solution of Fully Implicit Energy Conserving Primitive Equations

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Dietrich, *et al.* (1975) describe a procedure which has been successfully applied to a fully time-implicit model of the Navier-Stokes equations, using a vorticity-stream function formalism for two-dimensional incompressible flow in a channel. Since the space and time differenced barotropic vorticity equation used conserves enstrophy (mean squared vorticity) exactly, numerical stability is guaranteed. In fact, *the procedure was stable and accurate even when exceeding the advection CFL condition by a large factor.* The desirability of this for certain types of flow is well established.

In this note, an analogous fully time-implicit approximation of the Navier-Stokes primitive equations is presented and an analogous procedure for solving them is outlined. The difference equations conserve energy, mass, and momentum (discrete approximations) exactly (algebraically). Also, space and time truncation errors are small, as all difference approximations are centered over minimum intervals. The procedure outlined below for solving the coupled implicit equations is very similar to the one successfully used for the channel flow problem with a vorticity-stream function formalism*.

In the present model difference equations, inertia terms affect total energy only through boundary effects which vanish for a closed system. The procedure described for solving the fully implicit model equations determines one velocity component directly from the mass continuity equation and determines pressure from the corresponding momentum equation. (The required boundary pressure values are determined by the combined momentum, mass, and boundary constraints.) *No Poisson equation is solved explicitly for pressure,* although the calculated pressure satisfies the usual pressure Poisson equation.

The method is most easily described with reference to two-dimensional, incompressible, homogeneous, viscous flow confined to a square region, with rigid, non-slip boundary conditions. (The method generalizes to three-dimensional flow.) The mathematical statements of the constraints for such a flow in a region ($-1\leq x\leq 1$, $-1\leq y\leq 1$) may be written:

$$\frac{du}{dt}=-\frac{\partial\phi}{\partial x}+\nu\nabla^2 u \tag{1}$$

$$\frac{dv}{dt}=-\frac{\partial\phi}{\partial y}+\nu\nabla^2 v \tag{2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{3}$$

$$u=u(x,y,t),\ v=v(x,y,t),\ \phi=\phi(x,y,t)$$

$$u(1,y,t)=u(-1,y,t)=u(x,1,t)=u(x,-1,t)=0 \tag{4}$$

$$v(1,y,t)=v(-1,y,t)=v(x,1,t)=v(x,-1.t)=0 \tag{5}$$

$$u(x,y,0)=U_0(x,y),\ v(x,y.0)=V_0(x,y) \tag{6}$$

where ν is a specified diffusion coefficient and

$$\frac{d(\ )}{dt}=\frac{\partial(\ )}{\partial t}+\frac{\partial[u(\ )]}{\partial x}+\frac{\partial[v(\ )]}{\partial y}$$

Piacsek and Williams (1970) give a difference approximation of the nonlinear acceleration terms which would lead to exact energy conservation even if equation (3) were not satisfied exactly. However, the solution procedure to be described satisfies equation (3) exactly, so it is preferable to use the below difference approximations which

* The use of the vorticity-stream function approach in multiply connected problems would require use of certain primitive equation integrals to determine the pressure field and some of the stream function boundary values (which otherwise must be specified arbitrarily).

are more accurate and also conserve momentum.

The following time-implicit centered difference equations corresponding to Equations (1) through (3) conserve energy exactly when $\nu=0$. Letting

$$\bar{u}=\frac{1}{2}(u^t+u^{t+\Delta t}) \text{ and } \bar{v}=\frac{1}{2}(v^t+v^{t+\Delta t}),$$

the difference equations are (refer to the staggered grid illustrated by Figure 1):

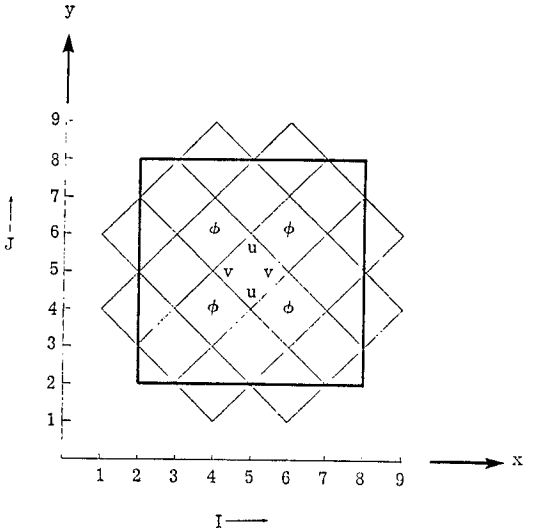

Fig. 1. Staggered grid for u, v, and ϕ. The $u(I, J)$ values are determined at even J and odd I, $v(I, J)$ are determined at odd J and even I, and $\phi\,(I, J)$ are determined for even I and even J.

$$\frac{u_{i,j}^{t+\Delta t}-u_{i,j}^{t}}{\Delta t}=-\frac{1}{4\Delta}\Big[\Big(\bar{u}_{i,j}+\bar{u}_{i+2,j}\Big)^2-\Big(\bar{u}_{i-2,j}+\bar{u}_{i,j}\Big)^2+\Big(\bar{u}_{i,j}+\bar{u}_{i,j+2}\Big)\Big(\bar{v}_{i-1,j+1}+\bar{v}_{i+1,j+1}\Big)-\Big(\bar{u}_{i,j-2}+\bar{u}_{i,j}\Big)\Big(\bar{v}_{i-1,j-1}+\bar{v}_{i+1,j-1}\Big)\Big]-\frac{1}{\Delta}\Big(\phi_{i+1,j}-\phi_{i-1,j}\Big)+\frac{\nu}{\Delta^2}\Big(\bar{u}_{i+2,j}+\bar{u}_{i,j+2}+\bar{u}_{i-2,j}+\bar{u}_{i,j-2}-4\bar{u}_{i,j}\Big) \quad (7)$$

$$\frac{v_{i,j}^{t+\Delta t}-v_{i,j}^{t}}{\Delta t}=-\frac{1}{4\Delta}\Big[\Big(\bar{u}_{i+1,j-1}+\bar{u}_{i+1,j+1}\Big)\Big(\bar{v}_{i,j}+\bar{v}_{i+2,j}\Big)-\Big(\bar{u}_{i-1,j-1}+\bar{u}_{i-1,j+1}\Big)\Big(\bar{v}_{i-2,j}+\bar{v}_{i,j}\Big)+\Big(\bar{v}_{i,j}+\bar{v}_{i,j+2}\Big)^2-\Big(\bar{v}_{i,j-2}-\bar{v}_{i,j}\Big)^2\Big]-\frac{1}{\Delta}\Big(\phi_{i,j+1}-\phi_{i,j-1}\Big)+\frac{\nu}{\Delta^2}\Big(\bar{v}_{i+2,j}+\bar{v}_{i,j+2}+\bar{v}_{i-2,j}+\bar{v}_{i,j-2}-4\bar{v}_{i,j}\Big) \quad (8)$$

$$u_{i+1,j}-u_{i-1,j}+v_{i,j+1}-v_{i,j-1}=0 \quad (9)$$

Energy conservation in Equations (7) through (9) is demonstrated by multiplying Equations (7) and (8) by $\bar{u}_{i,j}$ and $\bar{v}_{i,j}$, respectively, and summing over the entire grid. Internal contributions on the right hand side cancel algebraically when $\nu=0$, leaving only boundary conditions which vanish for a closed system.

Equations (7) through (9) are a coupled nonlinear set for three fields. They can be solved in a manner analogous to the one used by Dietrich, *et al.* (1975). Basically, the nonlinear terms are formally resolved into a linearized part, associated with a given (or known) basic (or initial) state, plus a nonlinear part, involving products of deviations from the basic state. The linearized part is solved directly, using the highly efficient "generalized sweepout method" or "GSM" (Hirota, *et al.*, 1970 and Roache, 1971). Within each time step, the nonlinear perturbation products are iterated until sufficient convergence is attained. The perturbation product convergence rate within a given time step serves as an indicator of whether or not to relinearize during the time step.

A key part of the GSM is to properly define the recursion sequence and boundary conditions. A proper procedure for the linearized system corresponding to Equations (7) through (9) (for the unknown future u, v and ϕ fields), given the perturbation products which serve as forcing functions and given the known "present" flow at time level t, is as follows, with reference to the 9×9 staggered grid in Figure 1.

(1) The boundary conditions are imposed.

$$u(3,2)=u(5,2)=u(7,2)=u(3,8)=u(5,8)=u(7,8)=$$
$$v(2,3)=v(2,5)=v(2,7)=v(8,3)=v(8,5)=v(8,7)=$$
$$u(1,4)+u(3,4)=u(1,6)+u(3,6)=u(7,4)+u(9,4)=$$
$$u(7,6)+u(9,6)=$$
$$v(4,1)+v(4,3)=v(6,1)+v(6,3)=v(4,7)+v(4,9)=$$
$$v(6,7)+v(6,9)=0$$

(2) Equation (9) is applied at points $(i, j)=(2,4)$,

(2,6), (8,4), (8,6), (4,2), (6,2), (4,8) and (6,8) to get:

$$u(3,4)-u(1,4)=u(3,6)-u(1,6)=u(9,4)-u(7,4)=$$
$$u(9,6)-u(7,6)=$$
$$v(4,3)-v(4,1)=v(6,3)-v(6,1)=u(4,9)-v(4,7)=$$
$$v(6,9)-v(6,7)=0$$

(3) The last eight conditions of (1) are combined with (2) to get:

$$u(1,4)=u(3,4)=u(1,6)=u(3,6)=u(7,4)=u(9,4)=$$
$$u(7,6)=u(9,6)=$$
$$v(4,1)=v(4,3)=v(6,1)=v(6,3)=v(4,7)=v(4,9)=$$
$$v(6,7)=v(6,9)=0$$

(4) $u(5,4)$, $\phi(4,2)$ and $\phi(6,2)$ are guessed or extrapolated from previous results.

(5) Equation (9) is applied at (4,4) and (6,4) to determine $v(4,5)$ and $v(6,5)$.

(6) Equation (8) is applied at (4,3) and (6,3) to determine $\phi(4,4)$ and $\phi(6,4)$.

(7) Equation (7) is applied at (5,4) to determine $u(5,6)$.

(8) Equation (9) is applied at (4,6) and (6,6) to determine $v(4,7)$ and $v(6,7)$.

(9) Equation (8) is applied at (4,5) and (6,5) to determine $\phi(4,6)$ and $\phi(6,6)$.

(10) Equation (7) is applied at (5,6) to determine $u(5,8)$.

Unless $u(5,4)$, $\phi(4,2)$, and $\phi(6,2)$ are guessed correctly in step (4), the values of $v(4,7)$ and $v(6,7)$, determined in step (8), and $u(5,8)$, determined in step (10), will not agree with their values assigned in steps (1) and (3). However, the relations used to generate $v(4,7)$, $v(6,7)$, and $u(5,8)$ from $u(5,4)$, $\phi(4,2)$, and $\phi(6,2)$ are all linear, so there exists a linear relation among the three "top" values and the three guessed values (for a given forcing function). The GSM takes advantage of this linearity. Formally, a homogeneous solution (for null forcing functions) is determined which, when added to the particular solution generated in steps (1) to (10), cancels any error at the upper boundary. The relation between the three required guess values of the homogeneous solution and its top values may be determined once-and-for-all, since the relation depends only on the linear operators of the linearized difference equations. It is determined by generating three homogeneous solutions for any three non-trivial triplets of $u(5,4)$, $\phi(4,2)$, and $\phi(6,2)$. (Since there is an arbitrary constant in the ϕ field, the two ϕ-values should not be equal when $u(5,4)=0$ in a given triplet.) Once this relation is determined, each linearized problem in the time-marching sequence may be solved with $O(n)$ operations, where n is the total number of grid points. No method can be significantly more efficient than this highly efficient GSM procedure.

It should be noted that in applying the above procedure to high resolution problems, high precision arithmetic is required. However, a modified procedure can be used to solve high resolution problems, while using relatively low precision arithmetic. This is done by dividing a high resolution mesh into submesh regions and applying the above procedure to each submesh region. Residuals on the submesh boundaries are reduced by an iterative relaxation procedure, called "block-implicit relaxation" or BIR, as described by Dietrich (1974) and Dietrich, *et al.* (1975).

The above ten step procedure is attractive from a physical point of view. Boundary pressure values are determined directly from internal flow patterns and appear closely linked to the normal velocity boundary condition. The assignment of pressure values on a rigid boundary to meet normal flow conditions corresponds directly to rigid boundary mechanics. Using the grid configuration in Figure 1, the proper boundary physics is modelled with very good accuracy.

Although the above procedure generalizes to three-dimensional flows, the GSM algorithm is much less efficient in three dimensions. It may be preferable to transform three-dimensional problems to sequential two-dimensional problems of the type described above. One way to do this would be to use an alternating direction plane-implicit formulation to solve successive three-dimensional block submeshes of high resolution three-dimensional problems. The boundary conditions for the blocks at the beginning of a time step could be determined by high order extrapolation of previous results.

The three-dimensional block size would be limited by the size of two-dimensional cross-section which can be solved directly (with the available

computer precision) using the GSM (referred to above). It might prove beneficial to use smaller blocks in order to take better advantage of BIR's highly efficient time extrapolation capability (Dietrich, 1974 and Dietrich, *et al.*, 1975), as this would increase the convergence rate of the alternating direction plane-implicit sweeps. Ideally, the blocks would be staggered from one three-dimensional sweep over all blocks to the next (if needed) within a time step.

Finally, as noted by Dietrich, *et al.* (1975), the above procedure using the GSM is readily applied to certain problems for which one-dimensional ADI equations are singular or nearly singular. It can also be applied iteratively to nonlinear steady state problems.

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完全にインプリシットでエネルギーを保存するプリミティヴ方程式の数値解法

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